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CS matrix rings over local rings

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Abstract

Complete characterization of CS matrix rings $M_n(R)$, $n > 1$, over local rings R is obtained. Application to group algebras is derived as a particular case of the main result.

Keywords: CS-rings; Finitely $\sum$-CS modules; Local rings; Semiperfect rings; Locally finite groups; Nilpotent groups

1. Introduction

A ring R is called a right CS-ring if every essentially closed right ideal is a direct summand of R. Such rings have been studied by several authors (cf. [2–9]). It is known that if R is a commutative integral domain then $M_2(R)$ is a right CS-ring if and only if R is a Prüfer domain [5, Corollary 12.10] and that if R is a local (noncommutative) domain then $M_n(R)$, $n > 1$, is a right CS-ring if and only if R is a valuation domain [1, Lemma 3.6]. In this paper we first show that the $n \times n$ matrix ring $(n > 1)$ over a local ring R is right CS if and only if R is right uniform and for every right ideal K of R and for every R-homomorphism $f : K \to R$ there exists $u \in R$ such that either $f = l_u$ or $l_u f = I_K$, where l_u is the left multiplication by u and I_K is the identity map on K (Theorem 3.5). If, in addition, the radical of R coincides with the right singular ideal, then $M_n(R)$, $n > 1$, is a right CS-ring if and only if R is a right selfinjective ring (Theorem 3.6).

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Theorem 3.9 shows that if R is a commutative noetherian local ring then $M_n(R)$, $n > 1$, is a right CS-ring if and only if the classical quotient ring, $Q_{\mathrm{cl}}(R)$, is a local QF-ring such that for all $a \in Q_{\mathrm{cl}}(R)$ either $a \in R$ or a is invertible and $a^{-1} \in R$. Lemma 3.3, which is also of independent interest, describes all uniform summands of the right R-module R^n and plays an important role in the proof of our main result. In Section 4 we apply our machinery, developed in Section 3, to the local CS group algebras and also to semiperfect group algebras of nilpotent groups. For local group algebra KG of any group G it is shown that $M_n(KG)$, $n > 1$, is a right CS-ring if and only if $\mathrm{char}(K) = p$ and G is a finite p-group.

2. Notation and definitions

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right unital. For any two right R-modules M and N, M is said to be N-injective if for any submodule L of N and any R-homomorphism $\phi : L \to M$ there exists an R-homomorphism $\psi : N \to M$ such that $\psi |_L = \phi$. A right R-module M is said be injective if M is N-injective for all right R-modules N. A submodule K of a right R-module M is said to be essential in M, denoted by $K \subset_{\mathrm{e}} M$, if for any nonzero submodule L of M, $K \cap L \neq 0$. M is called a CS (or extending) module if every submodule of M is essential in a direct summand of M, equivalently, if every closed submodule of M is a direct summand of M. M is called finitely $\sum$-CS if direct sum of finite number of copies of M is CS. M is called CS with respect to uniform submodules if every uniform submodule of M is essential in a direct summand of M, equivalently, if every uniform closed submodule of M is a direct summand of M. M is said to satisfy condition C_3 if for any two summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a summand of M. A CS module is called quasi-continuous if it satisfies C_3. It is known that if $M \times N$ is quasi-continuous then M and N are injective relative to each other.

A ring R is said to be right CS (or CS with respect to uniform right ideals) if the right R-module R is CS (resp. CS with respect to its uniform right R-submodules). R is called right selfinjective if R_R is injective. R is called a right valuation ring if for any two right ideals I and J either $I \subset J$ or $J \subset I$. Let S be an overring of R. The subset $\{1 = a_1, a_2, \ldots, a_n\}$ of S is said to be a normalizing basis of $S_R({}_RS)$ if $a_iR = Ra_i$, $1 \leqslant i \leqslant n$. S is called R-projective if for any S-module M and for any S-submodule N of M, if N is an R-summand of M then it is also an S-summand of M. For a ring R, $J(R)$ will denote the Jacobson radical of R and $Z_r(R)$, the right singular ideal $\{r \in R \mid rI = 0$ for some essential right ideal I of $R\}$ of R. For a nonempty subset X of a ring R, $\mathrm{r.ann}_R(X)$ ($\mathrm{l.ann}_R(X)$) will denote the right (left) annihilator of X in R. If X is the singleton $\{a\}$ then we write $\mathrm{r.ann}_R(X) = \mathrm{r.ann}_R(a)$ ($\mathrm{l.ann}_R(X) = \mathrm{l.ann}_R(a)$). For an element a of R, l_a will denote left multiplication by a.

A group G is called locally finite if every finitely generated subgroup of G is finite. For a group G, $O_p(G)$ will denote the maximal normal p-subgroup and $\omega(RG)$ will denote the augmentation ideal of the group ring RG. If H is a subgroup of G, we will write $\omega(H)$ to denote $\omega(RH)RG$.

3. Main results

Lemma 3.1 [5, Corollary 7.8]. *A right module over a ring R with finite uniform dimension is* CS *if and only if it is* CS *with respect to uniform submodules.*

Lemma 3.2 [5, Lemma 12.8]. *The matrix ring $M_n(R)$ over a ring R is right* CS *if and only if R^n is a* CS*-module as a right R-module.*

Lemma 3.3. *Suppose R is a local right* CS*-ring. A uniform right R-submodule U of R^n is a summand of R^n if and only if $U = (a_1, a_2, \dots, a_n)R$, where some $a_i = 1$.*

Proof. Since R is a local right CS-ring, R is a uniform right R-module. Let U be a uniform summand of the right R-module R^n. Then $R^n = U \oplus K$ for some right R-submodule K of R^n. Since R is local right uniform and U is uniform, by Krull–Schmidt Theorem, $U \simeq R$ as right R-modules. Let α be the isomorphism from R to U such that $1 \to (a_1, a_2, \dots, a_n) \in R^n$. Then $U = (a_1, a_2, \dots, a_n)R$. Consider the R-isomorphism $f : U \to R$ where $f = \alpha^{-1}$. Extend f to f^* from $U \oplus K = R^n$ to R by setting $f^* = f$ on U, and $f^* = 0$ on K. Now every homomorphism from R^n to R can be represented by a $n \times 1$ matrix with entries in R. Let

$$f^* = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Since $(a_1, a_2, \dots, a_n) \in U$ is the preimage of $1 \in R$ under f,

$$(a_1, a_2, \dots, a_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 1,$$

that is, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$. If $a_1, a_2, \dots, a_n$ are all in $J(R)$ then $1 \in J(R)$, a contradiction. Hence there exists i, $1 \leqslant i \leqslant n$, such that a_i is a unit. But then $U = (b_1, b_2, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_n)R$ where each $b_j = a_j a_i^{-1}$ as desired. Conversely, any right R-submodule U of the form $(a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)R$ is a summand of R^n because

$$(a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)R \oplus \left(\oplus \sum_{\substack{j=1 \\ j \neq i}}^{n} e_j R \right) = R^n,$$

where $e_j \in R^n$ is the row vector all of whose entries are 0 except the jth entry which is 1. □

Next we give a simple fact regarding finite linearly preordered sets.

Lemma 3.4. *Let $(S, \geqslant)$ be a finite set with linear preorder $\geqslant$. Then there exists $x \in S$ such that $x \geqslant s$ for all $s \in S$.*

Proof. The proof follows by induction. □

Theorem 3.5. *The following are equivalent for a local ring R.*

(1) $M_n(R)$, $n > 1$, *is a right* CS-*ring.*
(2) $M_2(R)$ *is a right* CS-*ring.*
(3) *R is right uniform, and for every right ideal K of R and for every R-homomorphism $f : K \to R$ there exists $u \in R$ such that either $f = l_u$ or $l_u f = I_K$.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (3) Since $M_2(R)$ is right CS, $R \times R$ is a CS right R-module. Thus R is right CS and hence right uniform. Let K be right ideal of R and let $f : K \to R$. Let $U = \{(x, f(x)) \mid x \in K\}$. Then $U_R \simeq K_R$. Thus U is uniform. Since $R \times R$ is a CS right R-module, U is essential in a summand S of $R \times R$. By Lemma 3.3, $S = (1, b)R$ or $(a, 1)R$. If $U \subset_{\mathrm{e}} (1, b)R$ then for every $x \in K$, there exist $r \in R$ such that $(x, f(x)) = (1, b)r$. Thus $x = r$ and $f(x) = br$. It follows that $f = l_b$. If $S \subset_{\mathrm{e}} (a, 1)R$ then for every $x \in K$ there exist $r \in R$ such that $(x, f(x)) = (a, 1)r$. Thus $x = ar$ and $f(x) = r$. Hence $x = af(x) = l_a f(x)$ for every $x \in K$, that is, $l_a f = I_K$. This proves (3).

(3) ⇒ (1) By Lemma 3.2 it is sufficient to prove that R^n is CS as a right R-module. Since R is right uniform, by Lemma 3.1, we only need to consider uniform right R-submodules of R^n. Let U be a uniform right R-submodule of R^n. Let π_i, be the canonical projection of R^n onto ith direct summand, let $K_i = \pi_i(U)$ and let f_i be the restriction of π_i onto U, $i = 1, 2, \ldots, n$. Clearly,

$$U = \{(f_1(x), f_2(x), \ldots, f_n(x)) \mid x \in U\} \tag{1}$$

and $\bigcap_{i=1}^n \ker(f_i) = 0$. Since U is uniform, there exists $1 \leqslant i \leqslant n$ such that $\ker(f_i) = 0$. Obviously f_i is an isomorphism. We may assume without loss of generality that there exists a positive integer $k \leqslant n$ such that $f_1, f_2, \ldots, f_k$ are isomorphisms whereas $\ker(f_j) \neq 0$ for all $j = k+1, k+2, \ldots, n$. Given $1 \leqslant i, j \leqslant k$, $f_i f_j^{-1} : K_j \to K_i$ is an isomorphism of R-modules. By our assumption there exists $a \in R$ such that either $f_i f_j^{-1} = l_a$ or $l_a f_i f_j^{-1} = I_{K_j}$. Therefore either $f_i = l_a f_j$ or $f_j = l_a f_i$. Now we introduce a linear preorder on $\{1, 2, \ldots, k\}$ as follows. We set $i \geqslant j$ if there exists $a \in R$ such that $f_j = l_a f_i$. Obviously this binary relation $\geqslant$ is transitive and reflexive since $f_i = l_1 f_i$. Therefore it is a linear preorder relation. By Lemma 3.4, the set $\{1, 2, \ldots, k\}$ has a maximum element. Let i be the maximum element of $\{1, 2, \ldots, k\}$. We may assume without loss of generality that $i = 1$. Therefore there exist $a_2, a_3, \ldots, a_k \in R$ such that $f_j = l_{a_j} f_1$, $j = 2, 3, \ldots, k$. Let $k+1 \leqslant r \leqslant n$. By our assumption there exist $a_r \in R$ such that either $f_r f_1^{-1} = l_{a_r}$ or

$l_{a_r} f_r f_1^{-1} = I_{K_1}$. The latter possibility is ruled out by the fact that $\ker(f_r) \neq 0$. We see that there exist $a_2, a_3, \ldots, a_n \in R$ such that

$$f_j = l_{a_j} f_1, \quad j = 2, 3, \ldots, n. \tag{2}$$

Thus from (1) and (2) we obtain

$$\begin{aligned} U &= \bigl\{\bigl(f_1(x), f_2(x), \ldots, f_n(x)\bigr) \mid x \in U\bigr\} \\ &= (1, a_2, a_3, \ldots, a_n) f_1(U) = (1, a_2, a_3, \ldots, a_n) K_1. \end{aligned}$$

Therefore $U \subset_{\mathrm{e}} (1, a_2, a_3, \ldots, a_n)R$ where $(1, a_2, a_3, \ldots, a_n)R$ is a direct summand of R^n by Lemma 3.3 and we are done. □

Theorem 3.6. *Suppose R is a local ring with $J(R) = Z_r(R)$. Then $M_n(R)$ is right* CS *for some $n > 1$ if and only if R is right selfinjective.*

Proof. Let $M_n(R)$ be right CS for some $n > 1$. By Lemma 3.2, R^n is CS as a right R-module. Consequently R is right CS and hence right uniform. To prove R is right selfinjective, let K be a nonzero right ideal of R and let $f : K \to R$ be an R-homomorphism. By Theorem 3.5, there exists $u \in R$ such that either $f = l_u$ or $l_u f = I_K$. If $l_u f = I_K$ then f is a monomorphism and $uf(a) = a$ for every $a \in K$. We show that u is invertible in R for otherwise $u \in J(R) = Z_r(R)$. Thus $\mathrm{r.ann}_R(u)$ is essential in R. Since K is nonzero, $f(K)$ is nonzero. Consequently $\mathrm{r.ann}_R(u) \cap f(K) \neq 0$. Let $0 \neq f(a) \in \mathrm{r.ann}_R(u) \cap f(K)$. Then $a = uf(a) = 0$, a contradiction because $f(a) \neq 0$. Hence $f(a) = u^{-1}a$ for every $a \in K$. Thus $f = l_{u^{-1}}$. This proves the result. □

Since for a right uniform local ring R with nil radical $J(R) = Z_r(R)$, we have the following corollary.

Corollary 3.7. *Suppose R is a local ring with nil radical. Then $M_n(R)$ is right* CS *for some $n > 1$ if and only if R is right selfinjective.*

We call a ring R right almost selfinjective if for any right ideal K of R and any R-homomorphism $f : K \to R$ there exists $a \in R$ such that either $f = l_a$ or $l_a f = I_K$.

We do not know whether for a local ring R, $M_n(R)$ $(n > 1)$ being right CS implies that $M_n(R)$ is also left CS. In particular, whether the condition that $M_n(R)$ is a right CS-ring implies that R is right-left uniform and right-left almost selfinjective. Theorem 3.9 characterizes local uniform right-left almost selfinjective rings with acc on right-left annihilators. Before proving the theorem we prove the following lemma.

Lemma 3.8. *Let R be a local right selfinjective ring and let A be a subring of R such that for any $a \in R$ either $a \in A$ or a is invertible with $a^{-1} \in A$. Then we have the following.*

(1) *A satisfies right-left Ore conditions and R is both right as well as left classical ring of quotients of A.*

(2) *R_A is an injective A-module.*
(3) *A is a local right uniform almost right selfinjective ring.*

Proof. (1) Let S be the set of all elements of A which are not left or right zero divisors in A and let $a \in S$. We first show that a is neither a left nor a right zero divisor in R. If $ax = 0$ for some $0 \neq x \in R$, then $x \notin A$ and so x is invertible in R forcing $a = 0$, a contradiction. Thus a is not a left zero divisor in R. Similarly a is not a right zero divisor in R. We now claim that a is invertible in R. Consider the right R-homomorphism $f : aR \to R$ given by the rule $f(ay) = y$ for all $y \in R$. Since R is right selfinjective, there exists $b \in R$ such that $f = l_b$ and so $ba = 1$. Therefore $(1 - ab)a = 0$. Since a is not a right zero divisor, we get $ab = 1$. Thus every element of S is invertible in R. By hypothesis, for every $r \in R$, either $r \in A$, or r is invertible in R and $r^{-1} \in A$. Therefore R is both left and right ring of fractions of A (see [14, p. 50]). It now follows that A satisfies both left and right Ore conditions and R is the two-sided classical ring of quotients of A (see [14, Proposition 1.4, p. 51]).

(2) By [14, Proposition 3.5, p. 57], both ${}_AR$ and R_A are flat modules. Since R_R is injective, R_A is an injective module (see [10, Corollary 3.6A]).

(3) Since R is a local right selfinjective ring, R is right uniform. As R is the classical ring of quotients of A, we conclude that A is also right uniform. To show that A is local, let J be the set of all elements of A which are not invertible in A. By [11, Theorem 19.1], it is enough to show that J is closed under addition. Let $a, b \in J$. Assume that $a + b$ is invertible in A. Set $u = a(a + b)^{-1}$ and $v = b(a + b)^{-1}$. Then $u, v \in A$ and $u + v = 1$. As R is local, we may assume that u is invertible in R. Setting $w = u^{-1}v$, we see that either $w \in A$ or $w^{-1} \in A$. In the former case we have $u^{-1} = 1 - u^{-1}v = 1 - w \in A$ and so $a^{-1} = (a + b)^{-1}u^{-1} \in A$, a contradiction. In the latter case $v^{-1} = 1 - w^{-1} \in A$ forcing $b^{-1} = (a + b)^{-1}v^{-1} \in A$, a contradiction again. Thus A is local ring.

To prove that A is right almost selfinjective, let K be a right ideal of A and let $f : K \to A$ be a right A-homomorphism. Since R is injective as a right A-module, we may assume that $f : R \to R$. As R is the classical ring of quotients of A, it is easy to see that f is an endomorphism of right R-modules and so there exists $a \in R$ such that $f = l_a$. If $a \in A$, then there is nothing to prove. If $a \notin A$, then a is invertible and $b = a^{-1} \in A$. Therefore $l_b f = I_K$. This completes the proof. □

Theorem 3.9. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is local left* (*or right*) *uniform, almost left and right selfinjective, and satisfies acc condition on left and right annihilators.*
(2) *R satisfies both left and right Ore conditions. Its two-sided classical ring of quotients $Q = Q_{\mathrm{cl}}(R)$ is a local* QF*-ring and for any $a \in Q$ either $a \in R$ or a is invertible in Q with $a^{-1} \in R$.*

Proof. (1) ⇒ (2) Let S be the set of all elements of R which are not left zero divisors. We claim that every element of S is not a right zero divisor. Indeed, let $a \in S$. Assume that $x_1 a = 0$ for some $0 \neq x_1 \in R$. Since $\mathrm{r.ann}_R(a) = 0$, the map $f_1 : aR \to x_1R$, given by the rule $f_1(ay) = x_1 y$, $y \in R$, is a well-defined homomorphism of right R-modules.

By our assumption there exists $x_2 \in R$ such that either $f_1 = l_{x_2}$ or $l_{x_2} f_1 = I_{aR}$. The latter possibility is ruled out by the fact that $f_1(a^2) = x_1 a = 0$. Therefore $f_1 = l_{x_2}$. Hence $x_2 a = l_{x_2}(a) = f(a) = x_1 \neq 0$. Also $x_2 a^2 = x_1 a = 0$. Thus $x_2 \in \mathrm{l.ann}_R(a^2) \setminus \mathrm{l.ann}_R(a)$. Continuing in this fashion, we shall construct a strictly increasing chain

$$\mathrm{l.ann}_R(a) \subset \mathrm{l.ann}_R\big(a^2\big) \subset \cdots \subset \mathrm{l.ann}_R\big(a^n\big) \subset \cdots,$$

contrary to our assumption. Therefore our claim is established.

Given $a \in S$ and $x \in R$, we claim that there exists $b \in S$ and $y \in R$ such that $bx = ya$. Consider the map $f : aR \to xR$ given by the rule $f(ay) = xy$, $y \in R$. Obviously f is a well-defined homomorphism of right R-modules. By our assumption there exists $z \in R$ such that either $f = l_z$ or $l_z f = I_{aR}$, that is, either $x = za$ or $a = zx$. In either case our assertion is trivially true.

We note that S satisfies the left Ore condition. By left-right symmetry, S satisfies the right Ore condition as well. Therefore, R has the classical ring of quotients Q.

Again let $a \in S$ and $x \in R$. Then as in the previous paragraph, either $x = za$ or $a = zx$ for some $z \in R$, that is, either $x \in Ra$ or $a \in Rx$. Equivalently, either $xa^{-1} \in R$ or $(xa^{-1})^{-1} = ax^{-1} \in R$. We note that for any $y \in Q$ either $y \in R$ or y is invertible and $y^{-1} \in R$. It now follows that for any set $P \subset Q$, $\mathrm{l.ann}_Q(P) = \mathrm{l.ann}_R(P)$ and $\mathrm{r.ann}_Q(P) = \mathrm{r.ann}_R(P)$. In particular, Q satisfies acc condition on left and right annihilators. Since R is a right uniform ring, Q is also a right uniform ring. Further, since R is a local ring, Q is also local. Moreover, every element of $J(Q)$ is a zero divisor and $J(Q) \subset R$.

We now claim that Q is right selfinjective. To prove the claim, let U be a right ideal of Q and let $f : U \to Q$ be a right Q-homomorphism. We show that f is given by the left multiplication. If $U = Q$ we are done. So let $U \neq Q$ and so $U \subset J(Q) \subset R$. Since every element of U is right zero divisor, $f(U)$ consists of right zero divisors and so $f(U) \subset J(Q) \subset R$. By our assumption there exists $a \in R$ such that either $f = l_a$ or $l_a f = I_U$. In the latter case the uniformity of the ring R implies that a is not a left zero divisor in R and so $a \in S$ is invertible in Q forcing $f = l_{a^{-1}}$. Thus either $f = l_a$ or $f = l_{a^{-1}}$, that is, f is given by the left multiplication, as desired. It now follows that Q is a QF-ring (see [14, Theorem 3.5, p. 277]).

(2) $\Rightarrow$ (1) Since any subring of a QF-ring satisfies the acc condition on left and right annihilators, the result follows from Lemma 3.8. □

Corollary 3.10. *Let R be a commutative noetherian local ring. Then $M_n(R)$, $n > 1$, is a right* CS*-ring if and only if the classical quotient ring, $Q_{\mathrm{cl}}(R)$, is local* QF *such that for all $a \in Q_{\mathrm{cl}}(R)$ either $a \in R$ or a is invertible and $a^{-1} \in R$.*

If R is a local ring then $Z_r(R) \subset J(R)$. If $Z_r(R) = 0$ then R is a domain and it is known that for a local domain R, $M_n(R)$, $n > 1$, is right CS if and only if R is a right and left valuation domain [1, Lemma 3.6]. In case $Z_r(R) = J(R)$ then $M_n(R)$ is right CS for some $n > 1$ if and only if R is right selfinjective (Theorem 3.6). We now provide an example of a local ring R such that $M_n(R)$ is right CS, but R is neither a domain nor right selfinjective.

Example 3.1. Let R be a right and left valuation domain, not necessarily commutative and let D be its right classical ring of quotients. Let

$$T = \left\{ \begin{pmatrix} a & d \\ 0 & a \end{pmatrix} \,\middle|\, a \in R,\ d \in D \right\} \subseteq S = \left\{ \begin{pmatrix} a & d \\ 0 & a \end{pmatrix} \,\middle|\, a, d \in D \right\}.$$

Obviously S is a local QF-ring and T is a subring of S such that for every $x \in S$ either $x \in T$ or $x^{-1} \in T$. According to Lemma 3.8, T is a local right uniform ring such that for any right ideal K of T and any right T-homomorphism $f : K \to T$ there exists $a \in T$ such that either $f = l_a$ or $l_a f = I_K$. Obviously T is not a domain and T is not right selfinjective because $J(T) \neq Z_r(T)$.

We now give another interesting application of Lemma 3.3.

Theorem 3.11. *Suppose R is a local right* CS*-ring with radical equal to the set of all zero divisors. Then $R \times R$, as a right R-module, satisfies C_3. In particular, if R is a local right* CS*-ring with nil radical then $R \times R$, as a right R-module, has C_3.*

Proof. It is sufficient to consider uniform summands of $R \times R$ as a right R-module. By Lemma 3.3, a uniform summand of $R \times R$ is either of the type $(1, a)R$ or $(a, 1)R$. To prove our assertion let S_1 and S_2 be uniform summands of R such that $S_1 \cap S_2 = 0$. Thus there exists a and b in R such that $S_1 = (1, a)R$ or $(a, 1)R$ and $S_2 = (1, b)R$ or $(b, 1)R$. First assume that if S_1 and S_2 have 1's in the same position, say, $S_1 = (1, a)R$ and $S_2 = (1, b)R$. Now if both a and b are in $J(R)$ then $\mathrm{r.ann}_R(a - b) \neq 0$. Hence there exists $r \in R$ such that $(a - b)r = 0$. It follows that $0 \neq (1, a)r = (1, b)r \in S_1 \cap S_2$, a contradiction. Hence one of a and b, say a, is not in $J(R)$. But then $S_1 = (1, a)R = (a^{-1}, 1)R$. We, therefore, only need to consider the case when $S_1 = (a, 1)R$ and $S_2 = (1, b)R$. We now show that in this case $S_1 \cap S_2 = 0$ if and only if $1 - ab$ is invertible. First let $1 - ab$ be invertible and let $(a, 1)r = (1, b)s \in S_1 \cap S_2$. Then $ar = s$ and $r = bs$. Thus $abs = s$, that is, $(1 - ab)s = 0$. Since $1 - ab$ is invertible we get $s = 0$. Thus $S_1 \cap S_2 = 0$. Conversely, if $1 - ab$ is not invertible then $1 - ab \in J(R)$. Thus there exists $0 \neq x \in R$ such that $(1 - ab)x = 0$. Hence $0 \neq (a, 1)bx = (1, b)x \in S_1 \cap S_2$. This proves the claim. Next we show that $S_1 \oplus S_2 = R \times R$. It is sufficient to show that $(1, 0)$ and $(0, 1)$ belong to $S_1 \oplus S_2$. Consider the relation $(1, 0) = (a, 1)x + (1, b)y$. Then $ax + y = 1$ and $x + by = 0$. These equations give $(1 - ab)y = 1$ and $x = -by$. Since $1 - ab$ is invertible, $y = (1 - ab)^{-1}$ and $x = -b(1 - ab)^{-1}$. It follows that $(1, 0) \in S_1 \oplus S_2$. Similarly $(0, 1) \in S_1 \oplus S_2$. This completes the proof. □

Remark 3.1. We note that for the ring R in Theorem 3.11 if $R \times R$ is also CS as a right R-module then $R \times R$ is quasi-continuous and so R is injective as a right R-module [15, Properties 41.20, p. 367]. This gives an alternative proof of Corollary 3.7.

Remark 3.2. Using an argument similar to the one in Theorem 3.11, it can be proved that if R is a local right CS-ring with nil radical then R^n, as a right R-module, has C_3 on uniform summands.

4. Applications to group algebras

In this section we give applications of the results obtained in the previous section to group algebras.

Lemma 4.1. *Let K be a field and G be any group. If the group algebra KG is local right* CS *then* $\operatorname{char}(K) = p$, *$G$ is a locally finite p-group, and the radical of KG is nil.*

Proof. Let KG be local right CS. Then $J(KG) = \varpi(KG)$. By [13, Lemma 1.13, p. 415], $\operatorname{char}(K) = p$ and G is a p-group. Also as KG has no nontrivial idempotents, KG is uniform.

We will prove that G is locally finite. Let $H = \langle h_1, h_2, \ldots, h_n \rangle$ be a finitely generated subgroup of G. Since G is a p-group, for each i with $1 \leqslant i \leqslant n$, $o(h_i) = p^{k_i}$ for some k_i. For each i, let $u_i = 1 + h_i + h_i^2 + \cdots + h_i^{p^{k_i - 1}}$. Since $u_i KG \neq 0$ for each i and KG is uniform, $\bigcap_{i=1}^n u_i KG \neq 0$. Let α be a nonzero element of $\bigcap_{i=1}^n u_i KG$. Then $(h_i - 1)\alpha = 0$ for each i. Consequently

$$\left(\sum_{i=1}^n KG(h_i - 1)\right)\alpha = 0.$$

Thus $0 \neq \alpha \in \mathrm{r.ann}(\omega(H))$. Hence H is a finite group, as desired. □

Theorem 4.2. *Let K be a field and G be any group such that the group algebra KG is local. The matrix ring $M_n(KG)$, $n > 1$, is a right* CS*-ring if and only if* $\operatorname{char}(K) = p$ *and G is a finite p-group.*

Proof. The proof follows from Corollary 3.7 once we observe that the radical of KG is nil and that KG is right selfinjective if and only if G is finite. □

We now consider semiperfect group algebras of nilpotent groups.

Theorem 4.3. *Let K be a field and G be a nilpotent group such that the group algebra KG is semiperfect. Then the following are equivalent.*

(1) *$M_n(KG)$, $n > 1$, is a right* CS*-ring.*
(2) *$M_2(KG)$ is a right* CS*-ring.*
(3) *G is finite.*

Proof. We only need to prove (2) ⇒ (3). Since G is nilpotent, $J(KG)$ is nilpotent. By [13, Theorem 1.5, p. 409] either $\operatorname{char}(K) = 0$ and G is finite or $\operatorname{char}(K) = p$, G is locally finite, and $[G : O_p(G)] < \infty$. We can assume that p does not divide $[G : O_p(G)]$. For if p divides $[G : O_p(G)]$ then taking the unique Sylow p-subgroup $\frac{N}{O_p(G)}$ of $\frac{G}{O_p(G)}$ we get a normal subgroup N of G such that $[G : N] < \infty$ and we can replace $O_p(G)$ with N. Since $O_p(G)$ is normal in G, KG is $KO_p(G)$-free with normalizing basis, say $\{1 = a_1, a_2, \ldots, a_n\}$.

Also because p does not divide $[G : O_p(G)]$, KG is $KO_p(G)$-projective [13, Lemma 2.2, p. 274]. Let $S = KG$ and $R = KO_p(G)$. Since $M_2(S) = M_2(KG)$ is right CS, $S^2 = S \times S$ is CS as a right S-module.

We show that R^2 is a CS as a right R-module. Observe that $S^2 = R^2a_1 + R^2a_2 + \cdots + R^2a_n$ and there exist automorphisms σ_i $(1 \leqslant i \leqslant n)$ of the ring R such that $a_i r = \sigma_i(r)a_i$. Let A be a closed R-submodule of R^2. First we prove that AS is closed R-submodule of S^2. Note that $AS = Aa_1 + Aa_2 + \cdots + Aa_n$. Let $x = x_1a_{k_1} + x_2a_{k_2} + \cdots + x_ua_{k_u}$ be in the closure of AS in S^2 where $x \notin AS$. We may assume without loss of generality that each $x_i \notin A$. Now there exists an essential right ideal E of R such that $0 \neq xE \subset AS$. But $xy = x_1\sigma_{k_1}(y)a_{k_1} + x_2\sigma_{k_2}(y)a_{k_2} + \cdots + x_u\sigma_{k_u}(y)a_{k_u}$ for every $y \in E$. Since $0 \neq xE \subset AS$ there exists i such that $0 \neq x_i\sigma_{k_i}(E) \subset A$. Because $\sigma_{k_i}(E)$ is essential right ideal of R and A is closed, $x_i \in A$, a contradiction. Hence AS is closed R-submodule of S^2. Since S is R-projective, AS is a closed S-submodule of S^2 [12, Proposition 1.1]. Consequently AS is a summand of S^2. Let $S^2 = AS \oplus B$. Writing A_0 for $Aa_2 + \cdots + Aa_n$ and S_0 for $R^2a_2 + \cdots + R^2a_n$, we have $R^2 \oplus S_0 = A \oplus (A_0 \oplus B)$. It follows that $R^2 = A \oplus ((A_0 \oplus B) \cap R^2)$ proving that A is a summand of R^2. This proves that R^2 is CS as a right R-module. But then $M_2(R)$ is right CS. Since $R = KO_p(G)$ is local, by Theorem 4.2, $O_p(G)$ is finite. Consequently G is a finite group. □

Note added in proof

(1) It has been pointed out to us that Theorem 3.5 can also be obtained from Lemma 8 in [Yoshitomo Baba, Mamabu Harada, On almost M-projectives and almost M-injectives, Tsukuba J. Math. 14 (1) (1990) 53–69].

(2) Theorem 4.3 has now been extended “to solvable groups and linear groups.”

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References

[1] S. Barthwal, S.K. Jain, P. Kanwar, S.R. López-Permouth, Nonsingular semiperfect CS-rings, J. Algebra 203 (1998) 361–373.

[2] A. Behn, Polycyclic group rings whose principal ideals are protective, J. Algebra 232 (2000) 697–707.

[3] G.F. Birkenmeier, J.Y. Kim, J.K. Park, A counterexample for CS-rings, Glasg. Math. J. 42 (2) (2000) 263–269.

[4] A.W. Chatters, C.R. Hajarnavis, Rings in which every complement right ideal is a direct summand, Quart. J. Math. 28 (1977) 61–80.

[5] N.V. Dung, D. Van Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Pitman, London, 1994.
[6] J.L. Gómez Pardo, P.A. Guil Asensio, Every $\sum$-CS module has an indecomposable decomposition, Proc. Amer. Math. Soc. 129 (2001) 947–954.
[7] D. Van Huynh, S.K. Jain, S.R. López-Permouth, On the symmetry of Goldie and CS conditions over prime rings, Proc. Amer. Math. Soc. 128 (2000) 3153–3157.
[8] S.K. Jain, P. Kanwar, S.R. López-Permouth, Nonsingular semiperfect CS-rings II, Bull. London Math. Soc. 32 (2000) 421–431.
[9] S.K. Jain, P. Kanwar, S. Malik, J.B. Srivastava, KD_∞ is a CS-algebra, Proc. Amer. Math. Soc. 128 (2) (2000) 397–400.
[10] T.Y. Lam, Lectures on Modules and Rings, Springer, 1999.
[11] T.Y. Lam, A First Course in Noncommutative Rings, 2nd Edition, Springer, 2001.
[12] M.M. Parmenter, P.N. Stewart, Excellent extensions, Comm. Algebra 16 (1988) 703–713.
[13] D.S. Passman, The Algebraic Structure of Group Rings, John Wiley, New York, 1977.
[14] B. Stenström, Rings of Quotients, Springer-Verlag, 1975.
[15] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, UK, 1991.